

On the Calderón-Zygmund lemma for Sobolev functions

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Abstract

We correct an inaccuracy in the proof of a result in [Aus1].

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We recall the lemma.

Lemma 0.1. *Let $n \geq 1$, $1 \leq p \leq \infty$ and $f \in \mathcal{D}'(\mathbb{R}^n)$ be such that $\|\nabla f\|_p < \infty$. Let $\alpha > 0$. Then, one can find a collection of cubes (Q_i) , functions g and b_i such that*

$$(0.1) \quad f = g + \sum_i b_i$$

and the following properties hold:

$$(0.2) \quad \|\nabla g\|_\infty \leq C\alpha,$$

$$(0.3) \quad b_i \in W_0^{1,p}(Q_i) \text{ and } \int_{Q_i} |\nabla b_i|^p \leq C\alpha^p |Q_i|,$$

$$(0.4) \quad \sum_i |Q_i| \leq C\alpha^{-p} \int_{\mathbb{R}^n} |\nabla f|^p,$$

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$$(0.5) \quad \sum_i \mathbf{1}_{Q_i} \leq N,$$

where C and N depend only on dimension and p .

This lemma was first stated in [Aus1] in \mathbb{R}^n . Then it appears in various forms and extensions in [Aus2] (same proof), [AC] (same proof on manifolds), [AM] (on \mathbb{R}^n but with a doubling weight), B. Ben Ali's PhD thesis [Be] and [AB] (The Sobolev space is modified to adapt to Schrödinger operators), N. Badr's PhD thesis [Ba] and [Ba1, Ba2] (used toward interpolation of Sobolev spaces on manifolds and measured metric spaces) and in [BR] (Sobolev spaces on graphs). The same inaccuracy can be corrected everywhere as below. The proof of the generalisation to higher order Sobolev spaces in [Aus1] can also be corrected with similar ideas.

The second equation tells that g is in fact Lipschitz continuous. There is a direct proof of this fact in N. Badr's thesis [Ba]. The proof proposed in [Aus1] is as follows:

Define $b_i = (f - c_i)\mathcal{X}_i$ where c_i are appropriate numbers and (\mathcal{X}_i) forms a smooth partition of unity of $\Omega = \cup Q_i$ subordinated to the cubes $(\frac{1}{2}Q_i)$ with support of \mathcal{X}_i contained in Q_i . For example, the choice $c_i = f(x_i)$ for some well chosen x_i or $c_i = m_{Q_i}f$, the mean of f over the cube Q_i , ensures that $\sum_i |b_i|\ell_i^{-1}$ is locally integrable (ℓ_i is the length of Q_i) and that $\sum_i b_i$ is a distribution on \mathbb{R}^n . Then g defined as $g = f - \sum_i b_i$ is a distribution on \mathbb{R}^n . Its gradient ∇g can be calculated as $\nabla g = (\nabla f)\mathbf{1}_F + h$ in the sense of distributions (on \mathbb{R}^n) with $h = \sum_i c_i \nabla \mathcal{X}_i$. It is then a consequence of the construction of the set $F = \Omega^c$ that $|\nabla f|$ is bounded on F by α and then it is shown that $|h|$ is bounded by $C\alpha$, which implies the boundedness of $|\nabla g|$.

Everything is correct in the argument above BUT the representation of h . The series $\sum_i c_i \nabla \mathcal{X}_i$, viewed as the distributional derivative on \mathbb{R}^n of $\sum_i c_i \mathcal{X}_i$, may not be a measurable function (section) on \mathbb{R}^n . For example, if $c_i = 1$ for all i , then $\sum_i \nabla \mathcal{X}_i = \nabla \mathbf{1}_\Omega$ is a non-zero distribution supported on the boundary of Ω (a measure if Ω has locally finite perimeter). One needs to renormalize the series to make it converge in the distribution sense.

Here we give correct renormalizations of h . A first one is obtained right away from differentiation of g :

$$h = - \sum_i (f - c_i) \nabla \mathcal{X}_i.$$

The convergence in the distributional sense in \mathbb{R}^n is in fact hidden of [Aus1].

A second one is

$$h = - \sum_m \left(\sum_i (c_m - c_i) \nabla \mathcal{X}_i \right) \mathcal{X}_m.$$

This representation converges in the distributional sense in \mathbb{R}^n and can be shown to be a bounded function.

Let us show how to obtain the second representation in the sense of distributions. Then the proof of boundedness is as in [Aus1]. Take a test function ϕ in \mathbb{R}^n . Then by definition the distribution $\sum_i \nabla b_i$ tested against ϕ is given by

$$\sum_i \int \nabla b_i \phi.$$

To compute this, we take a finite subset J of the set I of indices i and we have to pass to the limit in the sum restricted to J as $J \uparrow I$. Because now the sum is finite, and all functions have support in the set Ω , we can introduce $\sum_m \mathcal{X}_m = \mathbf{1}_\Omega$. We have

$$\sum_{i \in J} \int \nabla b_i \phi = \sum_m \sum_{i \in J} \int \nabla b_i \mathcal{X}_m \phi.$$

Now recall that $b_i = (f - c_i) \mathcal{X}_i$. Call I_m the set of indices such that the support of \mathcal{X}_i meets the support of \mathcal{X}_m . By property of the Whitney cubes, I_m is a finite set with bounded cardinal. Hence we can write

$$\sum_m \sum_{i \in J} \int \nabla b_i \mathcal{X}_m \phi = \sum_m \sum_{i \in J \cap I_m} \int \nabla f \mathcal{X}_i \mathcal{X}_m \phi + \sum_m \sum_{i \in J \cap I_m} \int (f - c_i) \nabla \mathcal{X}_i \mathcal{X}_m \phi.$$

It is clear that the first sum in the RHS converges to $\int_\Omega \nabla f \phi$ as $J \uparrow I$. As for the second it is equal to

$$\sum_m \sum_{i \in J \cap I_m} \int (c_m - c_i) \nabla \mathcal{X}_i \mathcal{X}_m \phi + \sum_m \sum_{i \in J \cap I_m} \int (f - c_m) \nabla \mathcal{X}_i \mathcal{X}_m \phi$$

As one can show (with the argument in [Aus1]) that

$$\sum_m \sum_i |c_m - c_i| |\nabla \mathcal{X}_i| |\mathcal{X}_m| \leq C\alpha$$

one has

$$\lim_{J \uparrow I} \sum_m \sum_{i \in J \cap I_m} \int (c_m - c_i) \nabla \mathcal{X}_i \mathcal{X}_m \phi = - \int h \phi$$

where h is defined above.

Finally, write

$$\sum_m \sum_{i \in J \cap I_m} \int (f - c_m) \nabla \mathcal{X}_i \mathcal{X}_m \phi = \sum_m \int b_m R_{m,J} \phi$$

with

$$R_{m,J} = \sum_{i \in J \cap I_m} \nabla \mathcal{X}_i.$$

By construction of the \mathcal{X}_i and properties of Whitney cubes,

$$\sum_{i \in I_m} |\nabla \mathcal{X}_i| \leq C \ell_m^{-1}$$

where ℓ_m is the length of Q_m , and on the support of \mathcal{X}_m

$$\sum_{i \in I_m} \nabla \mathcal{X}_i = \sum_{i \in I} \nabla \mathcal{X}_i = 0.$$

As $\sum_i |b_m| \ell_m^{-1}$ has been shown to be locally integrable, one can conclude by the Lebesgue dominated convergence theorem that

$$\lim_{J \uparrow I} \sum_m \int b_m R_{m,J} \phi = 0.$$

References

- [Aus1] P. Auscher. On L^p -estimates for square roots of second order elliptic operators on \mathbb{R}^n , *Publ. Mat.* **48** (2004), 159–186.
- [Aus2] P. Auscher, *On necessary and sufficient conditions for L^p estimates of Riesz transform associated elliptic operators on \mathbb{R}^n and related estimates*, *Memoirs Amer. Math. Soc* 186 (871) (2007).
- [AB] P. Auscher & B. Ben Ali, *Maximal inequalities and Riesz transform estimates on L^p spaces for Schrödinger operators with nonnegative potentials*, *Ann. Inst. Fourier* **57** no. 6 (2007), 1975-2013.

- [AC] P. Auscher & T. Coulhon, *Riesz transforms on manifolds and Poincaré inequalities*, Ann. Sc. Norm. Super. Pisa Cl. Sci. (5) **4** (2005), 1–25.
- [AM] P. Auscher & J.M. Martell, *Weighted norm inequalities, off-diagonal estimates and elliptic operators. Part I: General operator theory and weights*, Adv. Math. **212** (2007), no. 1, 225-276.
- [Ba] Nadine Badr, PhD thesis, University of Orsay, 2007.
- [Ba1] Nadine Badr, Real interpolation of Sobolev spaces, arXiv:0705.2216, to appear in Math. Scand.
- [Ba2] Nadine Badr, Real interpolation of Sobolev spaces associated to a weight, arXiv:0705.2268.
- [BR] N. Badr , E. Russ, Interpolation of Sobolev spaces, Littlewood-Paley inequalities and Riesz transforms on graphs, arXiv:0802.0922v1. To appear in Pub. Mat.
- [Be] Besma Ben Ali, PhD thesis, University of Orsay, 2008.